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Reduction in stages and complete quantization of the MIC–Kepler problem

Ivailo M Mladenov^{†§} and Vasil V Tsanov^{‡||}

[†] Institute of Biophysics, Bulgarian Academy of Sciences, Acad. G Bonchev Str., Bl. 21, Sofia 1113, Bulgaria

[‡] Department of Mathematics University of Sofia, 5 James Bourchier Bld, Sofia 1126, Bulgaria

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Abstract. The one-parameter deformation family of the standard Kepler problem known as the MIC–Kepler problem is completely quantized using the explicit momentum mapping of the torus actions on some toric manifolds and some equivariant cohomology theory. These manifolds appear as symplectic faces of the system during reduction process.

1. Introduction

Phase spaces of classical Hamiltonian systems are cotangent bundles of smooth configuration manifolds and their quantization does not present serious problems (see section 2). However, the first step—the prequantization—produces only part of the quantum numbers and one should use other devices in order to obtain the spectrum of a complete set of Dirac observables. Here we present a detailed treatment of a concrete dynamical system and show that reduction in stages (either at a classical or quantum level) produces the desired information about the spectrum. The system in question is the one-parameter deformation family of the Kepler problem known as the MIC–Kepler problem (see section 4). Despite that geometric quantization concept is thirty years old such treatment is absent even for the standard Kepler problem. A possible explanation of this situation can be traced back to the general fact that one can quantize unambiguously only functions which are polynomials up to a second degree in phase space coordinates while the square of the angular momentum which is a fourth-degree polynomial does not belong to this set. On the other hand, the choice of the momentum as an element of the complete set of observables is dictated by the spherical symmetry of the problem. It is well known that symmetries manifest themselves by separating the variables in the Schrödinger equation in appropriate coordinate systems and this is related to the existence of constants of motion. Simultaneous diagonalization of the Hamiltonian and the third components of the momentum and Runge–Lenz vector corresponds to a separation of variables in parabolic coordinates as noticed by Bargmann. Working in a much more abstract setting we will follow essentially the same idea in order to derive the missing quantum numbers. From a mathematical point of view the results which will be presented below follow from quantization of the momentum map associated with free torus actions on symplectic (toric) manifolds.

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2. Geometric quantization

2.1. Kostant–Souriau programme

On any symplectic manifold (M, ω) the symplectic form ω generates a Lie algebra structure in the space $R^\infty(M)$ of smooth real-valued functions on M . The problem of describing the representations of $R^\infty(M)$ was approached for the first time by Dirac [1] in the case $(M \equiv R^{2n}, \omega \equiv dp \wedge dq)$. It has been generalized by Segal [2] for phase spaces which are cotangent bundles. Finally, Kostant [3] and Souriau [4] treated arbitrary symplectic manifolds. They observed that if we are able to associate to every classical variable a quantum one, then the commutator of two quantum variables should represent up to a multiplicative number the Poisson bracket of the classical ones. This part of the programme nowadays is called prequantization. Below we summarize the relevant notions and definitions.

Definition 2.1. *The symplectic manifold (M, ω) is pre-quantizable if $[\omega/2\pi]$ is in the image of the map*

$$H_{Cech}^2(M, \mathbb{Z}) \rightarrow H_{deRham}^2(M, \mathbb{R}) \quad (2.1)$$

where $[\]$ denotes the de Rham cohomological class.

When M is a compact manifold this condition is equivalent to

$$\frac{1}{2\pi} \int_{\sigma} \omega \in \mathbb{Z} \quad \text{for every two-cycle } \sigma \in H_2(M, \mathbb{Z}). \quad (2.2)$$

It produces the quantization of charge, spin and energy levels of some physical systems.

If (M, ω) is pre-quantizable, then there exists a line bundle with a Hermitian form $h(\cdot, \cdot)$ $L \rightarrow M$, whose Chern class is $\frac{1}{2\pi}[\omega]$, equipped with a Hermitian connection ∇ whose curvature form is $-i\omega$ [3, 5–7]. The irreducibility of the representation which is the second stage (quantization) of the programme is achieved by also introducing a new structure called polarization. A real polarization on M is a map assigning to each point $m \in M$, a real subspace $F_m \subset T_m(M)$ which is maximally isotropic and integrable.

Example 2.2. *Let Q be a smooth manifold and let T^*Q be its cotangent bundle. If $\{p_i, q_i\}$ are local canonical coordinates in T^*Q , then an easy check shows that the vector fields*

$$X_1 = \frac{\partial}{\partial p_1}, X_2 = \frac{\partial}{\partial p_2}, \dots, X_n = \frac{\partial}{\partial p_n}$$

*define a real polarization over T^*Q which is known as vertical polarization.*

Example 2.3. *The two-dimensional sphere (S^2, ω) does not allow real polarization for any symplectic form ω , because of the non-existence of non-singular real vector field on S^2 .*

The last example suggests the generalization of the above notion, namely: a complex polarization over M is a map F assigning to each point $m \in M$ a subspace F_m of $T_m^{\mathbb{C}}(M)$ which defines a maximally isotropic integrable distribution such that the space $D_m = F_m \cap \bar{F}_m$ is of some fixed dimension κ (independent of the point $m \in M$). The polarization F is called Kählerian if $F_m \cap \bar{F}_m = 0$. For any kind of polarization F the potential θ of the symplectic form ω (i.e. $\omega = d\theta$) is adapted to it if $\theta(X) = 0$ for every $X \in F$. The quantum pre-Hilbert space is the linear space of the polarized sections of L whose definition is as follows: let M, ω, L, ∇ and F be as above. The space of smooth polarized sections of L is by definition

$$L^F = \{s \in \Gamma(L); \nabla_X s = 0, \text{ for all } X \in \mathfrak{X}(M, F)\}.$$

In order to define a scalar product L^F we need some measure (or rather density). To introduce it we consider the elements of the cotangent bundle $T^*(M)$ which vanish on F . They form a

subbundle $F^\circ \subset T_C^*(M)$ which is called the annihilator of F . By the definition of the symplectic form, the map

$$v \in F \rightarrow i(v)\omega \in F^\circ$$

is an isomorphism between F and F° . The line bundle $K_F = \wedge^n F^\circ$ on M will be further referred to as the canonical bundle of F . If $V = \{v_1, v_2, \dots, v_n\}$ is a (local) basis of F , then

$$K_V = i(v_1)\omega \wedge i(v_2)\omega \wedge \dots \wedge i(v_n)\omega$$

is a basis in K_F and for every $g \in GL(n, C)$, $(K)_{gV} = \det g \cdot K_V$.

Let (M, ω) be a symplectic manifold and let F is a complex polarization. We shall say that M is a metaplectic manifold if there exists a line bundle $N^{1/2}$ over M such that

$$N^{1/2} \otimes N^{1/2} = K_F.$$

One can show that (M, ω) is metaplectic if and only if the first Chern class of K_F is zero modulo two and this property does not depend on the choice of F . In this case the group $H^1(M, \mathbb{Z}_2)$ parameterizes the set of ‘square roots’, i.e. the set of all $N^{1/2}$ which satisfy the above condition. The sections of $N_F^{1/2}$ which are constant along F are called half-forms normal to F . The line bundle $\tilde{Q} = L^F \otimes N_F^{1/2}$ over M is called a quantum line bundle corresponding to the above data. We can introduce a scalar product in the space $\Gamma(\tilde{Q})$

$$\langle s_1, s_2 \rangle = \int_M s_1 \bar{s}_2 \quad (2.3)$$

and it is easy to see that the integrand is a density. Our Hilbert space \mathcal{H}^F is obtained by completing $\Gamma(\tilde{Q})$ with respect to the norm. The classical observables which can be quantized directly are the ones which preserve the polarization F , i.e. $\{f \in R^\infty(M); [X_f, F] \subset F\}$, where X_f is defined by the equation $i(X_f)\omega = -df$. If $\psi = s \otimes v$, where $\psi \in \Gamma(\tilde{Q})$, $s \in \Gamma(L^F)$, $v \in \Gamma(N_F^{1/2})$ are sections of the corresponding line bundles, the quantum operator associated with f acts in \mathcal{H}^F as follows:

$$\hat{f}(\psi) = (-i\nabla_{X_f} + f)s \otimes v - is \otimes \mathcal{L}(X_f)v. \quad (2.4)$$

Identifying the sections of L^F with functions on M (locally) the action of \hat{f} in \mathcal{H}^F can be written in the form

$$\hat{f}\psi = (-iX_f - \theta(X_f) + f)\varphi \otimes v - i\varphi \otimes \mathcal{L}(X_f)v \quad (2.5)$$

where θ is a local potential form of ω .

Actually, this explicit formula has found very few applications (cf section 6) as most of the treatments end with checking the consistency of the scheme relying on (2.4).

2.2. Cxyz–Hess scheme

Most components of the above scheme have an easy classical interpretation in the case when the symplectic manifold (M, ω) is a Kähler, i.e. when the manifold M has a complex structure

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \quad (2.6)$$

and a Hermitean metric g such that

$$\omega = \text{Im } g. \quad (2.7)$$

Using the notation of the previous section we can choose $F = T^{0,1}M$ and refer to formula (2.6) as giving a ‘Kählerian’ polarization. The determinant line bundle K of the annihilator $F^\circ \cong (T^{1,0}M)^*$ is in algebraic geometry the well known canonical bundle. The problem

which arises when $c_1(K)$ is not an even cohomological class as in the case of $\mathbb{C}P^{2n}$ because of the general result

$$c_1(\mathbb{C}P^n) = (n+1)\omega \quad (2.8)$$

is solved by Czyz [8] and Hess [9] via a slight modification of geometric quantization scheme which is outlined below.

Definition 2.4. Let (M, ω) be such Kählerian manifold that $[q] = \frac{1}{2\pi}[\omega] - \frac{1}{2}c_1(M)$ belongs to the image of $\epsilon : H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{R})$ and q is positive, i.e. $q(\sigma) \geq 0$ for any positively oriented two-cycle $\sigma \in H_2(M, \mathbb{R})$. A complex line bundle Q with $c_1(Q) = q$ is called a quantum bundle.

We fix a positive harmonic representative $\eta \in c_1(Q)$ and a Hermitian connection ∇ whose curvature is $-2\pi i\eta$. Now we also have a ∇ -invariant Hermitian structure $h_\eta(\cdot, \cdot)$ on Q . We recall that, the curvature of the hermitean metric h_η satisfies

$$\frac{i}{2\pi} \partial \bar{\partial} \log h_\eta \simeq \frac{\omega}{2\pi} - \frac{1}{2}c_1(M).$$

The space of holomorphic sections $H^0((M, Q))$ of Q is a Hilbert space \mathcal{H} with the scalar product

$$\langle s, t \rangle = \int_M h_\eta(s, t) \Omega_\eta \quad \omega_\eta = 2\pi \eta \quad s, t \in \Gamma(M, Q) \quad n = \frac{1}{2} \dim M$$

where $\Omega_\eta := \frac{(-1)^{n(n-1)/2}}{n!} \omega_\eta \wedge \omega_\eta \wedge \cdots \wedge \omega_\eta$ is the natural volume form on M . If our manifold M is simply connected the above Hermitian structure is determined up to a positive factor and \mathcal{H} is determined up to an isomorphism which depends on the choice of the connection ∇ . The representations are constructed following the prequantization recipe in where we put (Q, ω_η) in place of (L, ω) . To the classical observable f (a function on the phase space), there corresponds a quantum operator

$$\delta(f) \in \text{End } H^0(M, Q) \quad \delta(f)s \equiv (-i\nabla_{X_f} + f)s$$

where $s \in H^0(M, Q)$, and now the vector field X_f is defined by:

$$i(X_f)\omega_\eta = -df.$$

The only problem with this postulate is that ω_η is not always non-degenerate. More detailed exposition can be found in Czyz [8] and Hess [9].

3. Classical and quantum reductions

When a Lie group G acts symplectically (canonically) on the phase space (P, ω) of the Hamiltonian system (P, ω, H) leaving the Hamiltonian H invariant it generates quite naturally a mapping from P into the dual space \mathfrak{g}^* of its Lie algebra \mathfrak{g} whose components are integrals of motion for the dynamical system. The motion takes place inside a constraint submanifold $C \subset P$ and sometimes possesses gauge degrees of freedom. Passing on a new manifold where they are discarded has been known for centuries in mechanics as reduction procedure and its modern formulation given below is due to Marsden and Weinstein [10].

Theorem 3.1. Let (P, ω) be a symplectic manifold on which acts canonically the Lie group G , and $J : P \rightarrow \mathfrak{g}^*$ be the Ad^* -equivariant momentum mapping of this action. Let us suppose that $\mu \in \mathfrak{g}^*$ is a regular value of J and that the isotropy group G_μ act freely and properly on $J^{-1}(\mu)$. Then $P_\mu = J^{-1}(\mu)/G_\mu$ is a symplectic manifold with a symplectic form defined by

$\pi_\mu^* \omega_\mu = i_\mu^* \omega$ where $\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$ is the canonical projection and $i_\mu : P_\mu \rightarrow P$ is the embedding. Let $H : P \rightarrow \mathbb{R}$ is G -invariant Hamiltonian function. The flow induced on P_μ is also Hamiltonian with Hamiltonian function H_μ defined by the relation $H_\mu \circ \pi_\mu = H \circ i_\mu$.

If a Hamiltonian system (M, ω, H) allows another symmetry group action commuting with that of G , then the reduced system $(P_\mu, \omega_\mu, H_\mu)$ keeps this symmetry.

A special case of the above theorem which will be of immediate interest is the case when P is a cotangent bundle $T^*(M)$ of some manifold M on which acts freely and properly the one-parameter Lie group G . Let $M \rightarrow N = M/G$ be the induced principal G -bundle and let $\tilde{\alpha}$ be the connection one-form. The reduced symplectic manifold P_μ is symplectomorphic with T^*N whose symplectic form ω_μ is the sum of the canonical form on T^*N and a magnetic term $\mu \tau_N^* d\tilde{\alpha}$ where τ_N is the canonical projection $\tau_N : T^*N \rightarrow N$ [11].

Thus, each Hamiltonian system with symmetry can be treated as a dynamical system either on (P, ω) or (P_μ, ω_μ) and what is more important—there is no formal distinction at a classical level between working on the initial or reduced phase space. There are plenty of strong results concerning the quantum mechanical counterpart of this situation which tell us when quantization and reduction are coherent procedures (see [12–15]). In order to give the reader the flavour of what to expect in this situation and because we will make use of it we quote the following result.

Theorem 3.2 (Guillemin and Sternberg [13]). *Let us suppose that the (extended) phase space (P, ω) is a compact and quantizable, G is a compact Lie group, $0 \in \mathfrak{g}^*$ is a regular value of J and F is a Kählerian G -invariant polarization over P . Then, there exists an isomorphism between the G -invariant sections of L^F and the sections of the quantum line bundle over the reduced phase space (P_0, ω_0) .*

The situation is even more favourable—in the above setting the reduction and (pre)-quantization are interchangeable procedures.

4. The MIC–Kepler problem

The Hamiltonian system $(T^*\dot{R}^3, \Omega_\mu, H_\mu)$, where

$$\begin{aligned} T^*\dot{R}^3 &\equiv T^*(R^3 \setminus \{0\}) \equiv \{(p, q) \in R^3 \times R^3; q \neq 0\} \\ \Omega_\mu &= d\theta + \sigma_\mu \quad \theta = \sum_{j=1}^3 p_j dq_j \quad \sigma_\mu = -\mu/(2|q|^3) \sum_{i,j,k=1}^3 \epsilon_{ijk} q_i dq_j \wedge dq_k \quad (4.1) \\ H_\mu &= \frac{1}{2}|p|^2 - \alpha/r + \mu^2/2r^2 \quad |q|^2 = q_1^2 + q_2^2 + q_3^2 = r^2 \quad \alpha, \mu \in R \quad \alpha > 0 \end{aligned}$$

is known as the MIC–Kepler problem [16, 17]. In standard physical terminology, the problem consists in studying the motion of a charged particle in a field which is the superposition of a magnetic monopole field $\vec{B}_\mu = -\mu\vec{q}/|q|^3$, the fields generated by the Newtonian potential $-\alpha/r$ and a centrifugal potential $\mu^2/2r^2$. We will see that the energy level submanifolds $H_\mu^{-1}(E)$ for negative values of the energy are filled up with closed orbits. This hints at a presence of ‘hidden’ symmetry and ‘accidental’ degeneracy of the spectrum. Actually, the ‘hidden’ symmetry group of the Hamiltonian system $(T^*\dot{R}^3, \Omega_\mu, H_\mu)$ is $SO(4)$ generated by the constants of motion

$$\vec{L}^\mu = \vec{q} \times \vec{p} + \mu\vec{q}/r \quad \vec{A}^\mu = (\vec{L}^\mu \times \vec{p} + \alpha\vec{q}/r)/\sqrt{-H_\mu}$$

which may be interpreted as a ‘total angular momentum’ and a generalized Runge–Lenz vector. The names are borrowed from the standard Kepler problem which can be viewed as a special ‘point’ of this one-parameter deformation family. The classical Kepler problem ($\mu = 0$) was

geometrically quantized by Simms [18] and Mladenov and Tsanov [19]. Here we will apply geometric quantization to the extended and the reduced phase spaces of the Hamiltonian system $(T^*\dot{R}^3, \Omega_\mu, H_\mu)$ which results in coinciding spectra. We shall present them as follows.

Theorem 4.1 (Mladenov and Tsanov [17]). *The discrete spectrum (bound states) of the MIC–Kepler problem (α —fixed, μ —fixed and quantized) consists of energy levels :*

$$E_N = -\alpha^2/2N^2 \quad N = |\mu| + 1, |\mu| + 2, |\mu| + 3 \dots$$

The magnetic charge μ can take the values

$$\mu = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2 \dots$$

and the multiplicity of the energy level E_N is

$$m(E_N) = N^2 - \mu^2.$$

5. Conformal Kepler problem

The Hamiltonian system $(T^*\dot{R}^4, \Omega, H_\alpha)$, where

$$\begin{aligned} T^*\dot{R}^4 &= T^*(R^4 \setminus \{0\}) = \{(y, x) \in R^4 \times R^4; x \neq 0\} \\ \Omega &= dy \wedge dx = \sum_{j=1}^4 dy_j \wedge dx_j \end{aligned} \quad (5.1)$$

and

$$H_\alpha = (|y|^2 - 8\alpha)/8|x|^2 \quad \alpha\text{—fixed constant}$$

is known as the conformal Kepler problem [20]. Let us additionally introduce two other Hamiltonian functions on the phase space $(T^*\dot{R}^4, \Omega)$: the Harmonic oscillator,

$$K_\lambda = (|y|^2 + \lambda^2|x|^2)/2 \quad \lambda\text{—an arbitrary positive constant}$$

and the ‘momentum’,

$$M = \frac{1}{2}(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3).$$

Lemma 5.1. *Let $E < 0$ and $\lambda = \sqrt{-8E}$. Then*

$$H_\alpha^{-1}(E) = K_\lambda^{-1}(4\alpha)$$

and the flows defined by the Hamiltonians H_α and K_λ coincide on these hypersurfaces up to re-parameterization.

Proof. Taking into account the above definition it is obvious that we have

$$4|x|^2(H_\alpha + \lambda^2/8) = K_\lambda - 4\alpha$$

which proves the first statement. Further on H_α and K_λ will be denoted by H and K .

In order to prove the second one we need only to notice that the Hamiltonian vector fields X_H and X_K when restricted to energy level submanifolds $H^{-1}(E) = K^{-1}(4\alpha)$ are related as follows:

$$4|x|^2X_H = X_K$$

and so the proof of the lemma is complete. \square

The complex coordinates on $(T^*\dot{R}^4, \Omega)$ written below depend on the same arbitrary positive constant λ chosen above

$$\begin{aligned} z_1 &= \lambda(x_1 + ix_2) - i(y_1 + iy_2) & z_2 &= \lambda(x_3 + ix_4) - i(y_3 + iy_4) \\ z_3 &= \lambda(x_1 - ix_2) - i(y_1 - iy_2) & z_4 &= \lambda(x_3 - ix_4) - i(y_3 - iy_4). \end{aligned} \quad (5.2)$$

In these coordinates $T^*\dot{R}^4 \equiv \mathbb{C}^4 \setminus D$, where

$$D = \{z \in \mathbb{C}^4, z_1 = -\bar{z}_3, z_2 = -\bar{z}_4\}$$

and the symplectic form Ω is (up to a multiplicative constant) the standard Kähler form on \mathbb{C}^4

$$\Omega = \frac{i}{4\lambda} dz \wedge d\bar{z} = \frac{i}{4\lambda} \sum_{j=1}^4 dz_j \wedge d\bar{z}_j.$$

Finally, the Hamiltonian functions K and M may be written in these coordinates as

$$K = (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)/4 \quad (5.3)$$

and

$$M = (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)/8\lambda. \quad (5.4)$$

We remark that the Hamiltonians and the symplectic form Ω are well defined over the manifold

$$\dot{\mathbb{C}}^4 = \mathbb{C}^4 \setminus \{0\} \supset T^*\dot{R}^4.$$

Let K_t, M_t denote the flows of the Hamiltonian systems $(\dot{\mathbb{C}}^4, \Omega, K), (\dot{\mathbb{C}}^4, \Omega, M)$.

Lemma 5.2. *For every $z \in \dot{\mathbb{C}}^4$ and $s, t \in \mathbb{R}$, the corresponding flows are:*

$$K_t z = (e^{i\lambda t} z_1, e^{i\lambda t} z_2, e^{i\lambda t} z_3, e^{i\lambda t} z_4) \quad (5.5)$$

$$M_s z = (e^{is/2} z_1, e^{is/2} z_2, e^{-is/2} z_3, e^{-is/2} z_4). \quad (5.6)$$

In particular, the flows of all three Hamiltonians H, K and M commute where defined.

Proof. The explicit expressions for the flows K_t, M_s are obtained by direct calculations. The last assertion follows from these expressions and lemma 5.1.

Thus the flow M_s defines a symplectic $U(1)$ -action over $\dot{\mathbb{C}}^4$. The ‘momentum’ for this action is M itself. Let us remark that the set D and consequently its complementary set $T^*\dot{R}^4$ are invariant under this $U(1)$ -action. Through every point there passes one orbit and the Hamiltonian function H exactly invariant on these orbits. Hence, the Hamiltonian system $(T^*\dot{R}^4, \Omega, H)$ can be reduced with respect to $U(1)$. The result of this reduction is summarized in the following lemma. \square

Lemma 5.3 ([17, 20]). *Let $\mu \in \mathbb{R}$ be the value of the momentum map of the lifted Hopf action on $T^*\dot{R}^3$. Then*

$$M^{-1}(\mu)/U(1) \equiv T^*\dot{R}^3$$

and when reduced Ω and H produce Ω_μ and H_μ , i.e. one ends with the MIC–Kepler problem.

Besides, if one reduces the constants of motion of the conformal Kepler problem :

$$\begin{aligned} M_1 &= (z_1 \bar{z}_2 + z_2 \bar{z}_1 - z_3 \bar{z}_4 - z_4 \bar{z}_3)/8\lambda & A_1 &= (z_1 \bar{z}_2 + z_2 \bar{z}_1 + z_3 \bar{z}_4 + z_4 \bar{z}_3)/8 \\ M_2 &= (z_1 \bar{z}_2 - z_2 \bar{z}_1 + z_3 \bar{z}_4 - z_4 \bar{z}_3)/8 & A_2 &= (z_1 \bar{z}_2 - z_2 \bar{z}_1 - z_3 \bar{z}_4 + z_4 \bar{z}_3)/8 \\ M_3 &= (|z_1|^2 - |z_2|^2 - |z_3|^2 + |z_4|^2)/8 & A_3 &= (|z_1|^2 - |z_2|^2 + |z_3|^2 - |z_4|^2)/8 \end{aligned} \quad (5.7)$$

one gets the momentum \vec{L}^μ and the generalized Runge–Lenz vector \vec{A}^μ which are constants of the motion for the MIC–Kepler problem.

6. Quantization of the extended phase space

Definition 6.1. The level hypersurfaces of the map $J : \mathbb{C}^4 \setminus D \rightarrow \mathbb{R}^2$, $J(z) = (K(z), M(z))$ are called energy-momentum manifolds.

$$\mathcal{EM}(\lambda, \mu) = \{(y, x) \in T^*\dot{\mathbb{R}}^4; K = 4\alpha, M = \mu\}.$$

Under reduction $\mathcal{EM}(\lambda, \mu)$ falls down (via π_μ) over the energy hypersurface $H_\mu = -\lambda^2/8$ ($\lambda = \sqrt{-8E}$) of the MIC–Kepler problem. As a set $\mathcal{EM}(\lambda, \mu)$ is not empty if λ and μ satisfy the condition

$$\lambda|\mu| \leq 2\alpha.$$

In this section we assume (the reasons will be explained in the next one) that we have strong inequality $\lambda|\mu| < 2\alpha$. Following [21] (see also [22, 23]) we will change our viewpoint and will consider $(T^*\dot{\mathbb{R}}^4, \Omega)$ as an ‘extension’ of $(T^*\dot{\mathbb{R}}^3, \Omega_\mu)$.

We now prove theorem 4.1 in the context of the extended phase space.

Proof. We work with the complex coordinates defined in (5.2), the polarization F ‘spanned’ by the anti-holomorphic directions $\{\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, \frac{\partial}{\partial \bar{z}_3}, \frac{\partial}{\partial \bar{z}_4}\}$ and adapted potential $\theta = -\frac{i}{4\lambda} \bar{z} dz$ of Ω . The Hilbert space consists of ‘wavefunctions’ of the form $\psi = \varphi \otimes v$ where φ is holomorphic and

$$v = (dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4)^{1/2}.$$

Essentially Dirac’s quantization in the presence of constraints which are not eliminated at the classical level enforces them at the quantum level. In our case the constraints $K = 4\alpha$ and $M = \mu$ select the energy-momentum manifold $\mathcal{EM}(\lambda, \mu)$ and therefore the admissible quantum states are those which belong to the subspace \mathcal{H}_J of \mathcal{H} defined below:

$$\mathcal{H}_J = \{\psi \in \mathcal{H}; \hat{K}\psi = 4\alpha\psi, \hat{M}\psi = \mu\psi\}.$$

Taking into account all of the above and formula (2.5) we write down the quantized version of our operators as

$$\begin{aligned} \hat{K}\psi &= \lambda \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_4} + 2 \right) \varphi \otimes v \\ &= \lambda(\mathcal{N} + 2)\psi = 4\alpha\psi \quad \mathcal{N} = 0, 1, 2, \dots \end{aligned}$$

and

$$\hat{M}\psi = \frac{1}{2} \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - z_4 \frac{\partial}{\partial z_4} \right) \varphi \otimes v = \mu\psi$$

where φ is a homogeneous monomial of degree \mathcal{N} in z_1, z_2, z_3 and z_4 .

Introducing $N = \mathcal{N}/2 + 1$ and solving

$$2N\sqrt{-8E} = 4\alpha$$

we obtain the energy spectrum $E_N = -\alpha^2/2N^2$ as well

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 &= 2N - 2 \\ n_1 + n_2 - n_3 - n_4 &= 2\mu \quad n_i \geq 0 \quad i = 1, 2, 3, 4. \end{aligned}$$

The last constraint relation is equivalent to Dirac’s quantization of the magnetic charge:

$$\mu = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \dots$$

Besides we get:

$$n_1 + n_2 = N + \mu - 1 = \mathcal{N}_1 = 0, 1, 2, \dots$$

and

$$n_3 + n_4 = N - \mu - 1 = \mathcal{N}_2 = 0, 1, 2, \dots$$

which combined tell us that the admissible values of N are given by the formula $N = |\mu| + 1, |\mu| + 2, |\mu| + 3, \dots$

In order to find the degeneracies $m(E_N)$ one should notice that φ can be represented as a product $\varphi_1(z_1, z_2) \cdot \varphi_2(z_3, z_4)$ of homogeneous monomials of degree \mathcal{N}_1 and \mathcal{N}_2 respectively. So, the dimension of the Hilbert space $\mathcal{H}_{\mu, N}$ is:

$$m(E_N) = \dim \mathcal{H}_{\mu, N} = (\mathcal{N}_1 + 1)(\mathcal{N}_2 + 1) = N^2 - \mu^2$$

and this ends the proof of the theorem. \square

Remark 6.2. The Hilbert space $\mathcal{H}_{\mu, N}$ is the carrier space for the unitary irreducible representation $(\frac{\mathcal{N}_1}{2}, \frac{\mathcal{N}_2}{2}) = (\frac{N+\mu-1}{2}, \frac{N-\mu-1}{2})$ of the global symmetry group of the MIC–Kepler problem $\text{Spin}(4) \cong \text{SU}(2) \otimes \text{SU}(2)$ (μ -half-integer) or $\text{SO}(4)$ (μ -integer).

The wavefunctions in $\mathcal{H} = \oplus \mathcal{H}_{\mu, N}$ are labelled uniquely by four quantum numbers which are the eigenvalues of the complete set of commuting operators $\hat{M}(\mu), \hat{H}(N), \hat{M}_3(m), \hat{A}_3(\ell)$, where

$$\hat{M}_3 \psi = \frac{1}{2} \left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_4} \right) \psi = m \psi \quad (6.1)$$

and

$$\hat{A}_3 \psi = \frac{1}{2} \left(z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} - z_4 \frac{\partial}{\partial z_4} \right) \psi = \ell \psi. \quad (6.2)$$

From (6.1) and (6.2) we conclude immediately that m and ℓ can take either integer or half-integer values. We also remark that this result is closely related with convexity theorems about torus actions on symplectic manifolds [23]. Indeed, let us consider the flows U_ς, V_τ generated by M_3 and A_3 ,

$$U_\varsigma z = (e^{i\varsigma/2} z_1, e^{-i\varsigma/2} z_2, e^{-i\varsigma/2} z_3, e^{i\varsigma/2} z_4) \quad (6.3)$$

$$V_\tau z = (e^{i\tau/2} z_1, e^{-i\tau/2} z_2, e^{i\tau/2} z_3, e^{-i\tau/2} z_4) \quad (6.4)$$

in conjunction with K_t and M_s . Doing so we realize that we have at our disposal an action of the four-torus T^4 on our symplectic manifold (\mathbb{C}^4, Ω) . Introducing new ‘time’ variables

$$\begin{aligned} \phi_1 &= \lambda t + \frac{s}{2} + \frac{\varsigma}{2} + \frac{\tau}{2} \\ \phi_2 &= \lambda t + \frac{s}{2} - \frac{\varsigma}{2} - \frac{\tau}{2} \\ \phi_3 &= \lambda t - \frac{s}{2} - \frac{\varsigma}{2} + \frac{\tau}{2} \\ \phi_4 &= \lambda t - \frac{s}{2} + \frac{\varsigma}{2} - \frac{\tau}{2} \end{aligned} \quad (6.5)$$

this action takes the form

$$\Phi(\phi, z) = (e^{i\phi_1} z_1, e^{i\phi_2} z_2, e^{i\phi_3} z_3, e^{i\phi_4} z_4) \quad (6.6)$$

and the associated moment J_Φ is readily given by

$$J_\Phi(z) = \frac{1}{8\lambda} (|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2) \quad (6.7)$$

which makes it obvious that the image set is convex. Besides, our representation space is spanned by the homogeneous polynomials of degree \mathcal{N} in the variables z_1, z_2, z_3, z_4 on which the torus element $g = (e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4})$ is represented by the transformation

$$\sum a_n z^n \rightarrow \sum a_n e^{i\phi \cdot n} z^n. \quad (6.8)$$

The multi-indices $n = (n_1, n_2, n_3, n_4)$ which appear above obey $n_1 + n_2 + n_3 + n_4 = \mathcal{N}$ and provide labels for the irreducible multiplicity-free representations $\rho_{\mathcal{N}}$ of the torus T^4 .

7. Quantization of the orbit manifolds

In section 5 we have established that the energy level submanifolds consist entirely of closed orbits. This allows $H_\mu^{-1}(E)$ to be factorized with respect to the dynamical flow and the thus obtained manifold $H_\mu^{-1}(E)/U(1) = \mathcal{O}_\mu(E)$ is known as an orbit manifold. Its complete description as a symplectic manifold is given below.

Theorem 7.1 (Mladenov and Tsanov [17]). *Let $E < 0$ and $\lambda = \sqrt{-8E}$. Then*

- (i) *if $\lambda|\mu| < 2\alpha$, $\mathcal{O}_\mu(E) \cong P^1 \times P^1$*
- (ii) *if $\lambda|\mu| = 2\alpha$, $\mathcal{O}_\mu(E) \cong P^1$*
- (iii) *if $\lambda|\mu| > 2\alpha$, $H_\mu^{-1}(E) \equiv \emptyset$.*

The reduced symplectic form over $P^1 \times P^1$ is:

$$\Omega_\mu(E) = \frac{2\pi(2\alpha + \lambda\mu)}{\lambda}\omega_1 + \frac{2\pi(2\alpha - \lambda\mu)}{\lambda}\omega_2 \quad (7.1)$$

where

$$\omega_j = \frac{i}{2\pi} \frac{d\zeta_j \wedge d\bar{\zeta}_j}{(1 + |\zeta|^2)^2} \quad j = 1, 2 \quad (7.2)$$

and (ζ_1, ζ_2) are non-homogeneous coordinates on $P^1 \times P^1$.

The symplectic form over P^1 in item (ii) is the respective non-zero component of $\mathcal{O}_\mu(E)$ (depending on the sign of μ). This theorem reduces the quantization of the MIC–Kepler problem to a geometric quantization of the compact Kähler manifolds $P^1 \times P^1$ and P^1 . The proof is based on the following lemma.

Lemma 7.2. $\mathcal{O}_\mu(E) \cong J^{-1}(4\alpha, \mu)/U(1) \times U(1)$.

Proof. When $\mu \neq 0$ the orbits of the Hamiltonian H coincide with that of K described by lemma 5.1. In particular, none of them belongs to $\dot{\mathbb{C}}^4 \setminus T^*\dot{R}^4 = D \setminus \{0\}$ and therefore we have one-to-one correspondence between the orbits of the MIC–Kepler problem on the energy hypersurface $H_\mu = E$ and the orbits of the torus action on $J^{-1}(4\alpha, \mu)$ described in lemma 5.2 and this implies that the orbit spaces are identical.

What remains to be done in order to prove theorem 7.1 is to describe properly $J^{-1}(4\alpha, \mu)$. Now we remark that the system of equations $K = 4\alpha$, $M = \mu$ is equivalent to the system

$$|z_1|^2 + |z_2|^2 = 4(2\alpha + \lambda\mu) \quad |z_3|^2 + |z_4|^2 = 4(2\alpha - \lambda\mu)$$

so we can conclude that

$$J^{-1}(4\alpha, \mu) = \begin{cases} S^3 \times S^3 & \text{when } \lambda|\mu| < 2\alpha \\ S^3 & \text{when } \lambda|\mu| = 2\alpha \\ \emptyset & \text{when } \lambda|\mu| > 2\alpha. \end{cases}$$

The projection $p : S^3 \times S^3 \rightarrow P^1 \times P^1$ is defined through the Hopf's map of the corresponding factors

$$p(z_1, z_2, z_3, z_4) = ([z_1 : z_2], [z_3 : z_4])$$

where $[z_1 : z_2], [z_3 : z_4]$ are the homogeneous coordinates over $P^1 \times P^1$. In accordance with lemma 7.2 the projection p is just the factor-map

$$J^{-1}(4\alpha, \mu) \rightarrow J^{-1}(4\alpha, \mu)/(U(1) \times U(1)).$$

In this way item (i) of theorem 7.1 is proven. It is obvious that the restriction of p on the non-trivial factor gives the map we needed in order to prove (ii). Finally, item (iii) is trivial. It remains to compute the reduced symplectic form. In the non-homogeneous coordinates

$$(\zeta_1, \zeta_2) = (z_2/z_1, z_4/z_3)$$

on $P^1 \times P^1$, we have

$$p(z_1, z_2, z_3, z_4) = (\zeta_1, \zeta_2).$$

Referring to lemma 7.2 we can write

$$p^* \Omega_\mu(E) = \Omega_{|S^3 \times S^3}$$

where $S^3 \times S^3$ are spheres defined above. This is checked by an easy computation in coordinates and completes the proof of theorem 7.1. \square

Let us denote by ω_1 , respectively ω_2 , the pullbacks of the Fubini–Study forms of the first and second factors of the product $P^1 \times P^1$. Obviously the cohomology classes $[\omega_1]$, $[\omega_2]$ generate the group $H^2(P^1 \times P^1) = \mathbb{Z} \oplus \mathbb{Z}$ and

$$c_1(N_F^{1/2}) = -\frac{1}{2}c_1(P^1 \times P^1) = -([\omega_1] + [\omega_2]).$$

In view of the prequantization condition (2.2) we have

$$\frac{1}{2\pi} \Omega_\mu(E) = N_1 \omega_1 + N_2 \omega_2 \quad N_1, N_2 \in \mathbb{Z}$$

which means that

$$\begin{aligned} 2\alpha + \lambda\mu &= \lambda N_1 \\ 2\alpha - \lambda\mu &= \lambda N_2 \end{aligned}$$

as well as

$$\mu = \frac{1}{2}(N_1 - N_2) \quad \lambda = 4\alpha(N_1 + N_2).$$

Introducing $N = \frac{1}{2}(N_1 + N_2)$, we get immediately $N_1 = N + \mu$, $N_2 = N - \mu$ as well as the energy spectrum of the MIC–Kepler problem $E_N = -\alpha^2/2N^2$. The Hilbert space $H^0(P^1 \times P^1, Q_N)$ is non-trivial if the first Chern class of the line bundle $Q_N \rightarrow P^1 \times P^1$

$$c_1(Q_N) = (N_1 - 1)[\omega_1] + (N_2 - 1)[\omega_2]$$

is positive, i.e. $N_1, N_2 \geq 1$ and $N \geq |\mu| + 1$. Finally, the degeneracies $m(E_N)$ of the energy levels E_N which coincide with dimensionalities of the spaces of holomorphic sections of quantum line bundles Q_N are calculated by the Riemann–Roch–Hirzebruch theorem:

$$m(E_N) = \dim H^0(\mathcal{O}_\mu(E), Q_N) = N_1 N_2 = N^2 - \mu^2.$$

Remark 7.3. The observables M_3 and A_3 in the complete set which survive under reduction can be expressed in the nonhomogeneous coordinates (ζ_1, ζ_2) over $\mathcal{O}_\mu(E_N)$ as follows:

$$\begin{aligned} M_3^{\mu, N} &= \frac{N_1}{2} \frac{1 - |\zeta_1|^2}{1 + |\zeta_1|^2} - \frac{N_2}{2} \frac{1 - |\zeta_2|^2}{1 + |\zeta_2|^2} \\ A_3^{\mu, N} &= \frac{N_1}{2} \frac{1 - |\zeta_1|^2}{1 + |\zeta_1|^2} + \frac{N_2}{2} \frac{1 - |\zeta_2|^2}{1 + |\zeta_2|^2}. \end{aligned}$$

The expression for the third component of the Runge–Lenz vector is actually the momentum mapping of the circular action around vertical axes of the spheres. If we fix its value to be ℓ then the momentum manifold

$$N_1 \frac{|\xi_1|^2}{1 + |\xi_1|^2} + N_2 \frac{|\xi_2|^2}{1 + |\xi_2|^2} = \frac{N_1}{2} + \frac{N_2}{2} - \ell = N - \ell \quad (7.3)$$

is either the sphere S^3 when $N - \ell > 0$, four points when $N - \ell = 0$ or the empty set in the case $N - \ell < 0$. This can be seen quite easily if we introduce the following set of coordinates:

$$\xi_1 = \left(\frac{N_1}{1 + |\xi_1|^2} \right)^{1/2} \zeta_1 \quad \xi_2 = \left(\frac{N_2}{1 + |\xi_2|^2} \right)^{1/2} \zeta_2 \quad (7.4)$$

in which (7.3) becomes obviously

$$|\xi_1|^2 + |\xi_2|^2 = N - \ell. \quad (7.5)$$

In the first of the above listed cases we have a free action of $SO(2)$ on $J^{-1}(\ell)$ and therefore we can factorize it. The reduced manifold is topologically the sphere S^2 and the reduced symplectic form is

$$\omega_\ell = 2\pi(N - \ell)\sigma \quad (7.6)$$

where σ is the form (7.2) written in any of the non-homogeneous coordinates on the projective line $[\xi_1 : \xi_2]$. Now the quantization condition reads

$$(N - \ell)\sigma - \sigma = k\sigma \quad k \geq 0 \quad (7.7)$$

from which follows that the maximal value of ℓ is $N - 1$. Using the Riemann–Roch theorem one can easily find that the number of the global holomorphic sections of the reduced quantum bundle L_k over the sphere S^2 is $k + 1 = N - \ell$. Introducing $\xi := \xi_1/\xi_2$ the last function $M_3^{\mu, N}$ from the complete set of observables can be written as

$$(N - \ell) \frac{1 - |\xi|^2}{1 + |\xi|^2} + \mu \quad (7.8)$$

while the corresponding ‘quantum’ operator is

$$-2\xi \frac{\partial}{\partial \xi} + N - \ell - 1 + \mu. \quad (7.9)$$

The spectrum of this operator in $\Gamma(S^2, \mathcal{O}(L_k))$ is the finite set $\{-k + \mu, -k + \mu + 2, \dots, k + \mu - 2, k + \mu\}$. At the classical level (7.8) is just the momentum map of the circle action around the third axis of the sphere S^2 (so we can forget the additive constant μ) and if $|m| < N - \ell$ this action is free. The inverse image of the momentum map is a circle and after reduction we end with a point as reduced phase space. The representation space associated with this point is one-dimensional as the only $SO(2)$ -invariant section which descends from S^2 is the constant section.

Remark 7.4. *Since the first days of quantum mechanics the volume in the phase space was expected to be related to the number of pure states. This was proven to be asymptotically true (up to universal factor) by Heckman on the basis of the Duistermaat–Heckman exact stationary phase formula [24]. This formula involves the set of fixed points of the action which we have not considered yet. The S^1 action on S^2 was treated in [25] and the result (in our notation) is*

$$\text{vol}(S_m^2) = \begin{cases} 1 & \text{if } |m| < N - \ell & \text{a point} \\ 0 & \text{if } |m| \geq N - \ell & \text{empty.} \end{cases} \quad (7.10)$$

The S^1 diagonal action on $S^2 \times S^2$ which has four fixed points mentioned above is studied by Wu [26] and in that case

$$\text{vol}((S^2 \times S^2)_\ell) = \begin{cases} 2\pi(N - \ell) & \text{if } N - \ell > 0 \\ 0 & \text{if } N - \ell \leq 0. \end{cases} \quad (7.11)$$

Finally the volume of the orbit manifold $\mathcal{O}_\mu(N)$ is $4\pi^2 N_1 N_2$ and all this coincides with the results we have obtained before.

Remark 7.5. *Thus in our case there is complete coherence of results obtained at all levels, starting with the extended and ending with a point, i.e. the reduction–quantization technique is the straightforward formalism for the treatment of systems with high symmetries. The quantum numbers can be derived by quantizing any of the symplectic manifolds which appear at different stages of the reduction procedure.*

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